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A NON-PAC FIELD WHOSE MAXIMAL PURELY INSEPARABLE EXTENSION IS PAC

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ABSTRACT

We use the Mordell conjecture for function fields in order to construct a non-PAC field whose maximal purely inseparable extension is PAC.

A field K is **pseudo algebraically closed** (abbrev. **PAC**) if each absolutely irreducible variety defined over K has a K-rational point. A theorem of Ax and Roquette [FJ, Cor. 10.7] says that if K is PAC, then so is each algebraic extension of K. The converse is obviously false. However, Geyer and Jarden [GJ, Prob. 12.4] ask if the converse is at least true for purely inseparable extensions.

In this note we answer also the latter question negatively, by constructing a non-PAC field which has a purely inseparable PAC extension.

All fields in this note have a fixed positive characteristic p.

LEMMA 1: *Let $F = K(x_1, \ldots, x_n)$ be a finitely generated extension of a field K. Suppose that K is algebraically closed in F. Then*

$$\bigcap_{k=1}^{\infty} K(x_1^{p^k}, \ldots, x_n^{p^k}) = K. \qquad (1)$$

Proof: Denote the left hand side of (1) by F_0. Suppose first that K is perfect. Thus $K^p = K$. Hence $F_0 = F_0^p$ is also perfect. In addition F_0, as a subfield of F, is finitely generated over K [Lan, p. 64]. If F_0 were transcendental over K, we

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could choose a transcendental basis $t_1, \dots, t_r$ with $r \geq 1$. Then F_0 would have a finite degree over $E = K(t_1, \dots, t_r)$. On the other hand, since F_0 is perfect, $E(t_1^{1/p^m})$ would be contained in F_0 and would have degree p^m over E for each positive integer m. This contradiction proves that F_0 is algebraic over K. As K is algebraically closed in F, conclude that $F_0 = K$.

In the general case the maximal purely inseparable extension L of K is a perfect field. Hence, by the preceding paragraph,

$$F_0 \subseteq F \cap \bigcap_{k=1}^{\infty} L(x_1^{p^k}, \dots, x_n^{p^k}) = F \cap L = K.$$

So, $F_0 = K$. ∎

Suppose that F is a transcendental extension of K. We say that a curve C which is defined over F is **nonconstant**, if it is not birationally equivalent over $\tilde{K}F$ to a curve which is defined over $\tilde{K}$. Here $\tilde{K}$ is the algebraic closure of K.

PROPOSITION 2: *Let F be a finitely generated regular extension of a field K_0. Let C be a nonconstant curve of genus at least 2 which is defined over F. Then $C(F)$ is a finite set.*

Proof: This is the analog of the Mordell conjecture for function fields. See [Sam, pp. 80, 107, and 118]. ∎

LEMMA 3: *Let K be a finitely generated regular transcendental extension of a field K_0. Let C be a nonconstant curve over K of genus at least 2 and let F be a finitely generated regular extension of K. Then K has a finitely generated extension $E \subseteq F$ such that F/E is a finite purely inseparable extension and $C(K) = C(E)$.*

Proof: Let $F = K(x_1, \dots, x_n)$, and for each k write $F_k = K(x_1^{p^k}, \dots, x_n^{p^k})$. By Lemma 1, the intersection of all F_k is K. By Proposition 2, $C(F)$ is a finite set. Hence, there exists a positive integer k such that $C(F_k) = C(K)$. So, $E = F_k$ and F satisfy the assertion of the Lemma. ∎

THEOREM 4: *Let K_0 be a countable field of characteristic p and let K be a finitely generated transcendental extension of K_0. Then K has countable regular extensions $E \subseteq F$ such that F/E is a purely inseparable extension, E is not PAC, but F is PAC.*

Proof: Choose a nonconstant curve C of genus at least 2 which is defined over K (e.g., if $p \neq 2$ and $t \in K$ is transcendental over K_0, then $Y^2 = X^5 + t$ defines a nonconstant curve over K of genus 2; if $p = 2$, then $Y^2+Y = X+\frac{1}{X}+\frac{1}{X-1}+\frac{1}{X-t}$ defines a nonconstant curve over K of genus 3). By induction we construct two ascending towers of fields $K = E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ and $F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots$ and for each positive integer m we enumerate the absolutely irreducible varieties which are defined over E_m in a sequence, $V_{m1}, V_{m2}, V_{m3}, \ldots$ such that

(2a) E_m and F_m are finitely generated regular extensions of K,

(2b) F_m/E_m is a finite purely inseparable extension,

(2c) $C(E_m) = C(K)$, and

(2d) $V_{ij}(F_m) \neq \emptyset$ for $i, j = 1, \ldots, m$.

Indeed, suppose that $E_1, \ldots, E_{m-1}$, $F_1, \ldots, F_{m-1}$ and V_{ij} for $i < m$ and all j have been defined such that they satisfy (2). Let V be the direct product of V_{ij} for $i, j = 1, \ldots, m-1$. It is an absolutely irreducible variety defined over E_{m-1}. Let $\mathbf{x}$ be a generic point of V over E_{m-1}. Then $E'_m = E_{m-1}(\mathbf{x})$ is a finitely generated regular extension of E. Apply Lemma 3 to C, E_{m-1}, and K_0 and construct an extension $E_m \subseteq E'_m$ of E_{m-1} such that E'_m/E_m is a finite purely inseparable extension and $C(E_m) = C(E_{m-1})$. By (2c) for $m-1$ we have $C(E_m) = C(K)$. As E'_m is a regular extension of E_{m-1} it is linearly disjoint from $\tilde{K}F_{m-1}$ over E_{m-1}. Hence, $F_m = E'_m F_{m-1}$ is linearly disjoint from $\tilde{K}F_{m-1}$ over F_{m-1}. Since F_{m-1} is linearly disjoint from $\tilde{K}$ over K, we have that F_m is linearly disjoint from $\tilde{K}$ over K. Thus F_m is a regular extension of K. By construction, F_m is a finite purely inseparable extension of E_m and $V_{ij}(F_m) \neq \emptyset$ for $i, j = 1 \ldots, m$.

Let $E = \bigcup_{m=1}^{\infty} E_m$ and $F = \bigcup_{m=1}^{\infty}$. Then E and F are countable regular extensions of K. Also, F is purely inseparable over E. Hence, in order to prove that F is PAC it suffices, by a theorem of Roquette [FJ, Thm. 10.4], to prove that each absolutely irreducible variety V defined over E has an F-rational point.

Indeed, if V is such a variety, then $V = V_{ij}$ for some i and j. Let $m = \max\{i, j\}$. By (2d), V has an F_m-rational point, which is, of course, an F-rational point.

Finally, each point of $C(E)$ belongs to $C(E_m)$ for some m and therefore, by (2c), to $C(K)$. Thus $C(E) = C(K)$ is a finite set. Hence, by Rabinovich's trick [FJ, Prop. 10.1], E is not PAC. ∎

Note the maximal purely inseparable extension of E contains F and is therefore also PAC:

COROLLARY 5: *There exists a non-PAC field E whose maximal purely inseparable extension is PAC.*

Note that if F/E is a finite purely inseparable extension, then there exists m such that $F^{p^m} \subseteq E$. If F is PAC, then so is F^{p^m} (the fields are isomorphic) and therefore, by Roquette's theorem, also E.

PROBLEM 6: *Does there exist a purely inseparable extension of fields F/E of finite transcendence degree over $\mathbb{F}_p$ such that E is not PAC but F is?*

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